Coupled Superconductors

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I. INTRODUCTION AND BASIC FORMULAS

In this paper, I wish to describe the clarification which has taken place in our ideas on the nature and behavior of tunneling supercurrents since the original work was done on the subject, and to mention some recent experimental work. Tunneling supercurrents originally emerged out of a complicated calculation of tunneling currents as terms which did not go to zero when the voltage across the barrier was put equal to zero.[1] It is now clear, however, that they are of essentially the same nature as the familiar kind of supercurrent. The link between the two is provided by the Ginzburg–Landau theory,[2] and I shall first give a brief summary of this theory.

The first attempts to describe the superconducting phase transition, such as the Gorter–Casimir two-fluid model,[3] used a *real* order parameter, the density of superconducting electrons. These theories are based on an assumed dependence of the free energy on the order parameter, whose value is found by minimizing the free energy with respect to it. This type of theory explains the existence of a critical magnetic field and the second order phase transition in zero field, but cannot, without additional assumptions, attempt to give a description of normal-superconducting phase boundaries or the penetration region in a magnetic field. The Ginzburg–Landau theory describes these in a natural way. Unlike the two-fluid model it uses a *complex* order parameter ψ, interpreted as the wave function of the superconducting electrons. Thus $|\psi|^2$ takes the place of the density of superconducting electrons in the two-fluid model. The other new feature is a term in the free energy representing the kinetic energy of the superconducting electrons; this is where the complex nature of ψ enters. The kinetic energy term involves both ψ and the electromagnetic vector potential, and therefore couples together the electrons and the magnetic field. It thus turns out to be a natural consequence of the Ginzburg–Landau theory that the thermodynamically stable state of a superconductor in a magnetic field is a current carrying one. Since the Ginzburg–Landau theory, microscopic theories of superconductivity have been developed, giving an explanation of its assumptions in so far as they apply,[4] but as they do not add anything essentially new to the main discussion we shall ignore them except in Sec. V.

After these preliminaries, let us return to tunneling between superconductors. In the first place let us consider a system consisting of two superconductors completely isolated from each other. Like the Schrödinger equation, the Ginzburg–Landau equations have the feature that from any given solution others may be obtained by changing the phase of ψ. The complete isolation of our superconductors from each other implies that it must be possible to alter the phases of ψ in each independently. On the other hand in a single (simply connected) superconductor under given external conditions ψ is completely determined apart from a phase factor, so that all phase differences are fixed. If now we imagine the two superconductors to be separated by a barrier of normal metal or insulator whose thickness is gradually reduced to zero, it is reasonable to suppose that the properties of the system go over continuously from those of two isolated superconductors to those of a single superconductor. This must happen in the following way: the free energy of the system must contain a term which depends on the relative phases of the values of ψ on the two sides of the barrier and increases in magnitude as the barrier becomes thinner. When the barrier is thick the coupling energy is very small and the phases are free to change arbitrarily, but once the coupling energy becomes large compared with kT, the phases become effectively locked together. We assume the coupling free energy to be expressible as an integral over the barrier:

$$F = \int_S f(P)dS\,, \tag{1.1}$$

where $f(P)$ depends only on the nature of the barrier and the local values of ψ on the two sides of it (and possibly on temperature). Throughout this paper we

[1] B. D. Josephson, Phys. Letters **1**, 251 (1962). Owing to a computational error the value of j_1 given in this paper is incorrect. For the correct value see Ref. 24.

[2] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

[3] See, for example, D. Shoenberg, *Superconductivity* (Cambridge University Press, 1960), Chaps. 3 and 6.

[4] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958), [English transl.: Soviet Phys.—JETP **7**, 505 (1958)] and Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [English transl.: Soviet Phys.—JETP **9**, 1364 (1959)].

shall be concerned only with small fields and currents, which do not appreciably affect the density of superconducting electrons $|\psi|^2$, and so it will be sufficient to regard the surface free energy as a function of ϕ, the difference between the phases of ψ on the two sides of the barrier.[5]

Associated with the surface coupling energy is the possibility of supercurrents through the barrier. In fact the two are intimately related:

$$j = \frac{2e}{\hbar}\,\frac{\partial f}{\partial \phi}\,, \tag{1.2}$$

where j is the supercurrent through unit area of barrier and Gaussian units are used. Equation (1.2) is reminiscent of some formulas of classical thermodynamics, and can indeed be proved by classical thermodynamical means.[6] Anderson[7] has given a quantum mechanical derivation. Owing to the definition of ϕ as a phase difference, j and f must clearly be periodic functions of ϕ. The microscopic theory[1] shows that, in the limit of weak coupling, only first harmonics enter:

$$j = j_1 \sin\phi\,, \tag{1.3}$$

$$j = -\frac{\hbar}{2e} j_1 \cos\phi + \text{constant}\,. \tag{1.4}$$

(1.3) and (1.4) are good approximations for the barriers normally used in tunneling experiments.

It is sometimes convenient to use a gauge in which there is a large component of vector potential in the barrier normal to it; in this case, a gauge term,

$$-\,(2e/\hbar c)\int_1^2 \mathbf{A}\cdot d\mathbf{s}$$

must be added to ϕ, the integral being along a curve joining the superconductors.[8]

II. EQUILIBRIUM PROPERTIES

In this section, we shall assume that the coupling across the barrier is sufficiently strong to maintain a definite phase relationship between the two sides.[9] There is reason to believe that in this case the tunneling supercurrents are nondissipative and of essentially the same nature as ordinary supercurrents. For example, in a closed ring containing a barrier there is a set of different solutions to the modified Ginzburg–Landau equations, in which the phase of ψ changes by different multiples of 2π in going round the ring. Jumping from one solution to the other involves overcoming the coupling energy, and if this is impossible the higher energy solutions, which are current carrying ones, will be metastable.

Magnetic fields can pass between two superconductors separated by a barrier, in the form of a thin sheet of flux penetrating a short distance into the superconductors in the usual way. In the presence of flux ϕ varies over the barrier in a way determined by the following rule.[10] Consider two points P and Q on opposite sides of the barrier (and flux sheet) joined by two curves crossing the barrier at different points A and B. The difference between the phases of ϕ at A and B is proportional to the flux between the two curves, one flux quantum $hc/2e$ corresponding to a phase difference of 2π. This effect is closely related to the one predicted by Ehrenberg and Siday[11] for interference between electron beams in the presence of a magnetic field.

In differential form we may write

$$\text{grad}\,\phi = \frac{2ed}{\hbar c}\,(\mathbf{H} \times \mathbf{n})\,. \tag{2.1}$$

Here $\mathbf{H}$ is the field in the barrier, $\mathbf{n}$ is a unit vector normal to the barrier, and d is the effective thickness of the sheet of flux:

$$d = \lambda_1 + \lambda_2 + t\,, \tag{2.2}$$

where λ_1, λ_2 are the penetration depths on the two sides, and t is the barrier thickness.[12]

Combining (2.1) and (1.3) and Maxwell's equations, we obtain [10,13]

$$\nabla^2\phi = (\lambda')^{-2}\sin\phi\,, \tag{2.3}$$

where

$$\lambda' = (\hbar c^2/8\pi e j_1 d)^{\frac{1}{2}}\,. \tag{2.4}$$

Here ∇^2 is a two-dimensional operator. λ' is typically of the order of 1 mm. and has the significance of a penetration depth, as can be seen by putting $\sin\phi \sim \phi$ in (2.3).

The detailed behavior of coupled superconductors depends on the transverse dimensions of the barrier

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[5] Sign conventions: if the sides of the barrier are labeled 1 and 2 and $\mathbf{n}$ is a unit vector pointing from 1 to 2 then $\phi = \arg\psi_2 - \arg\psi_1$, $j = \mathbf{j}\cdot\mathbf{n}$, $V = V_1 - V_2$.

[6] The proof is similar to that used to find the energy stored in an inductor, and is based on Eq. (3.2).

[7] P. W. Anderson, *Proceedings of the Ravello Spring School*, 1963 (to be published).

[8] P. W. Anderson and J. M. Rowell, Phys. Rev. Letters **10**, 230 (1963).

[9] This will be discussed in more detail in Sec. IV.

[10] P. W. Anderson (private communication).

[11] W. Ehrenberg and R. E. Siday, Proc. Phys. Soc. (London) **B62**, 8 (1949). In the present case the flux quantum is half that in the case of interference of electron beams, since electron pairs are involved (Sec. III).

[12] Note that d is not simply the barrier thickness as has sometimes been supposed.

[13] R. A. Ferrell and R. E. Prange, Phys. Rev. Letters **10**, 479 (1963).

relative to λ'. Barriers large compared with λ' behave in a similar manner to superconductors of the second kind.[14] In weak magnetic fields diamagnetic currents screen out the field from the barrier except at points within a distance of the order of λ' from the edge. At a critical field[15]

$$H_c' = (32\hbar j_1/\pi e d)^{\frac{1}{2}} \tag{2.5}$$

of the order of 1 G, a second order phase transition occurs and quantized flux lines start to penetrate into the barrier, their separation decreasing as the field is increased. This is the ideal thermodynamic behavior; in practice it would appear that hysteresis may occur, owing to the attachment of flux lines to the edge of the barrier.[15] In the case considered by Ferrell and Prange,[13] a field of $\frac{1}{2}\pi$ times the critical field H_c' is required before flux starts penetrating. Owing to screening, when a supercurrent is passed through a barrier it is normally all carried in the neighborhood of the edges.[10] Ferrell and Prange[13] have shown that the effective current-carrying width of a barrier is $2\lambda'$. However, if a barrier is inhomogeneous, it may be possible for current to be carried by its interior, owing to flux pinning, just as in type II superconductors.[16,17]

The behavior of a barrier small compared with λ' is somewhat simpler. Just as with very thin films, the magnetic field inside such a barrier is almost constant. The main feature of interest is the field dependence of the critical supercurrent. As we have seen, the effect of a field is to cause ϕ to vary in phase over the barrier. Therefore, in view of (1.3), in a sufficiently strong field the barrier is split up into regions in which the current through it has opposite signs, and the maximum value of the total supercurrent is thus greatly reduced. The field dependence of the critical current is given by a Fraunhofer diffraction pattern formula for an aperture of the same shape as the barrier (and transmission amplitude j_1).[15] For a rectangular barrier with the field along a side, one obtains

$$I_c \propto \left|\frac{\sin\,(H/H_0)}{H/H_0}\right|, \tag{2.6}$$

where $H_0/2\pi$ is the field for which the barrier encloses one flux quantum $hc/2e$. Behavior of this type has been observed by Rowell,[18] and this experiment, together with the previous one of Anderson and Rowell,[8] indicates strongly that the currents observed are indeed tunneling through the barrier and not going through isolated metallic bridges.

III. NONEQUILIBRIUM PROPERTIES

In dealing with the nonequilibrium properties of coupled superconductors, the first requirement is to find a formula for the time dependence of ϕ. Here I give a not very rigorous derivation, which serves, however, to introduce the important topic of the interaction of tunneling supercurrents with photons.[19]

So far we have been using what may be called the wave picture of a superconductor—we have been dealing with ψ, the wave function of the superconducting electrons. From the microscopic theory of superconductivity it is known that the particles associated with the wave function are bound pairs of electrons called Cooper pairs.[4,20] In some ways a superconductor can be thought of as a Bose–Einstein condensation of Cooper pairs. The supercurrents we have been discussing may be viewed as Cooper pairs tunneling through the barrier [Fig. 1(a)]. If there is a

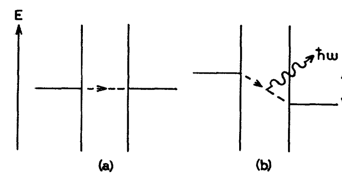

FIG. 1. Quasi-particle picture of tunneling supercurrents. (a) dc supercurrents at zero voltage. (b) ac supercurrents at nonzero voltage, giving rise to the emission of photons. Corresponding to each emitted photon a Cooper pair tunnels across the barrier in the direction of the applied voltage, conservation of energy requiring that $\hbar\omega = 2eV$. In the presence of a microwave field this process and its reverse can occur, giving rise to photon-induced supercurrents.

nonzero potential difference V between the two sides of the barrier, Cooper pairs on different sides of the barrier have energies differing by $\Delta E = 2eV$ (the charge of a pair being $2e$). Tunneling through the barrier can then take place only as a virtual process. We know from elementary quantum mechanics that in such a system there are oscillating currents at a frequency $\nu = \Delta E/h$, i.e.,

$$\nu = 2eV/h\,. \tag{3.1}$$

When the interaction between the oscillating currents and the electromagnetic field is taken into account, one finds that real processes can take place,

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[14] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].
[15] B. D. Josephson (unpublished work).
[16] P. W. Anderson, Phys. Rev. Letters **9**, 309 (1962).
[17] For example, a superconducting bridge across the barrier would have this effect, and a region with a high local value of f would behave similarly.
[18] J. M. Rowell, Phys. Rev. Letters **11**, 200 (1963).

[19] A more rigorous proof is given in reference 1. An alternative proof based on gauge invariance has been given by Anderson (reference 7).
[20] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).

energy being conserved by the emission of a photon [Fig. 1(b)]. In this case the radiation is coherent, since every photon comes from an identical process.

Taking into account the fact that the oscillating currents arise from changes in ϕ, we see that

$$\dot{\phi} = 2\pi\nu = 2eV/\hbar\,. \tag{3.2}$$

Now let us consider what happens if we apply to the barrier radiation of the same frequency as the oscillating supercurrents. When two oscillating systems are coupled together, energy can be transferred in either direction, depending on the phase relationships. The energy transfer process in this case is Fig. 1(b), or the reverse process.[21] Therefore, in the presence of microwave radiation dc supercurrents (i.e. the transfer of Cooper pairs across the barrier) can occur provided the potential difference is such that energy can be conserved by absorption or stimulated emission of a photon (multiphoton processes can also occur). This process is characterized by the appearance of constant voltage regions in the I–V characteristic, and this has recently been observed by Shapiro,[22] who made the interesting observation that for particular values of microwave power the specimen would spontaneously jump on to such a constant voltage region. Shapiro's specimen is therefore behaving as an ideal zero-impedance voltage source, powered by the microwave field.

For the supercurrent–photon coupling to be strong it is necessary that both the supercurrents and the electric field in the barrier produced by the microwave radiation should be in phase all over the barrier. Hence small barriers are favorable to the observation of the effect.

I shall now pass on to a different type of nonequilibrium effect. This concerns what happens when one applies rf fields to a system in which both sides of the barrier are originally at the same potential. If the rf fields are small enough they do not break up phase coherence across the barrier but merely cause oscillations in phase. As may be seen by taking the time derivative of (1.3) and using (3.2), the barrier behaves with respect to supercurrents exactly like an inductance $L_0 \sec\phi$ per unit area, where

$$L_0 = \hbar/2ej_1\,. \tag{3.3}$$

One must also take into account the capacitance C per unit area (as indeed one must to treat correctly the earlier work in this section). The following dispersion equation is obtained,[15] assuming dc fields and currents to be absent.[23]

$$(\omega/\omega_0)^2 = (k\lambda')^2 + 1\,, \tag{3.4}$$

where

$$\omega_0 = (L_0C)^{-\frac{1}{2}}, \tag{3.5}$$

which corresponds to frequencies of a few Gc/sec. When $\omega > \omega_0$, traveling waves localized in the neighborhood of the barrier can be propagated along it. These could be excited by applying alternating voltages across the barrier at its edge. When $\omega < \omega_0$, k is imaginary, and disturbances excited in this manner decay exponentially with distance from the edge, just as in the dc limit considered in Sec. II. The impedance across unit length of the edge of a semi-infinite barrier is $4\pi d\omega/kc^2$ (inductive when k is imaginary),[15] which is normally very small ($<10^{-3}$ Ω cm) except when $\omega \sim \omega_0$, when the impedance becomes theoretically infinite. When $\omega = \omega_0$, $k = 0$ and the disturbance is a type of plasma oscillation, with the current and electric field normal to the barrier and the magnetic field absent. The power that could be transmitted along a barrier without breaking up phase coherence should be sufficient to permit direct demonstration of the existence of the cutoff frequency ω_0.

A final point of interest in connection with this type of effect concerns quasi-particle currents. According to the microscopic theory,[1] the barrier conductivity which determines these is phase dependent:

$$\sigma = \sigma_0 + \sigma_1 \cos\phi\,. \tag{3.6}$$

The phase dependent term does not affect the dc I–V characteristics of the barrier because $\cos\phi$ averages out to zero, but it can affect the quasi-particle damping of waves of the type just considered.

IV. THERMAL AND ZERO-POINT FLUCTUATIONS

For very small barriers the only mode of oscillation important in producing fluctuations in ϕ is the $k = 0$, $\omega = \omega_0$ mode of Sec. III, in which the oscillations in ϕ are in phase all over the barrier. Assuming fluctuations in ϕ are small the oscillations can be assumed to be simple harmonic, and we obtain for the mean square deviation of ϕ:

$$\overline{(\Delta\phi)^2} = \frac{2e\omega_0}{j_1A}\left[\frac{1}{2} + \frac{1}{\exp(\hbar\omega_0/kT) - 1}\right], \tag{4.1}$$

where A is the area of the barrier. As the barrier is made larger, however, modes with $k \neq 0$ become important and eventually $\overline{(\Delta\phi)^2}$ must become inde-

[21] Similar processes for tunneling quasi-particles have been observed by A. H. Dayem and R. J. Martin [Phys. Rev. Letters **8**, 246 (1962)] and discussed theoretically by P. K. Tien and J. P. Gordon [Phys. Rev. **129**, 647 (1963)].

[22] S. Shapiro, Phys. Rev. Letters **11**, 80 (1963).

[23] In general both j_1 and λ' may be frequency dependent, the latter through the frequency dependence of λ.

pendent of A. Clearly in order to observe effects which depend on having phase coherence over the barrier, one must have $\overline{(\Delta\phi)^2} \ll 1$. This sets a lower limit to the value of j_1. As pointed out by Anderson and Rowell,[8] the temperature seen by the barrier is not necessarily the cryostat temperature, since thermal noise is transmitted to the specimen down the leads to the cryostat from room temperature circuitry. Since this is at a temperature of the order of a hundred times greater, it is clearly desirable to choose conditions so that this effect is minimized. To see the conditions necessary for this, consider the idealized circuit of Fig. 2. The specimen is represented by a damped tuned circuit and the room temperature source by a noise voltage V_N in series with a resistor R_1. It is seen that the object of reducing the current through L can be achieved by

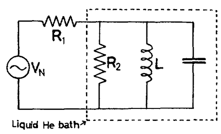

FIG. 2. Simplified circuit diagram illustrating the effects of electrical noise transmitted down the leads to the cryostat.

(i) increasing the source impedance R_1;

(ii) inserting an electrical filter cooled to liquid helium temperature in the leads to the specimen. In particular, frequencies of ω_0 or above where there may be high Q resonances should be removed;

(iii) if possible, damping out the resonances by reducing R_2, i.e., connecting a normal metal shunt across the barrier.

V. MICROSCOPIC THEORY

Various methods have been used to calculate j_1 from basic theory.[1,7,24,25] Many of these use the tunneling Hamiltonian method of Cohen *et al.*[26] The simplest method is that of Anderson,[7] who uses statistical mechanical perturbation theory to calculate f and then derives j_1 from (1.2). Formulas for j_1 have been given for absolute zero by Anderson[7] and calculated numerically for finite temperatures by Ambegaokar and Baratoff.[24] This method cannot be used for nonequilibrium processes, which can be dealt with by adiabatic perturbation theory.[1] An important result of the microscopic theory is that j_1 is proportional to the tunneling probability for quasi-particles, so that tunneling supercurrents are similar in magnitude to quasi-particle currents at voltages of the order of the energy gap.

[24] V. Ambegaokar and A. Baratoff, Phys. Rev. Letters **10**, 486 (1963) and **11**, 104 (1963) (erratum).

[25] P. G. de Gennes, Phys. Letters **5**, 22 (1963).

[26] M. H. Cohen, L. M. Falicov, and J. C. Phillips, Phys. Rev. Letters **8**, 316 (1962).

Objections have been raised against the tunneling Hamiltonian method on the grounds that it treats the barrier as a mathematical plane and ignores events taking place in it.[27] A method which avoids this difficulty is to describe propagation through the barrier by Green's functions.[15,25] This method has the additional advantages that it shows precisely the conditions necessary for coupling to occur, it indicates the relation between supercurrents through barriers and the usual kind of supercurrent, and it is not restricted to insulating barriers. Gor'kov's theory[4] allows one to express the current density in a superconductor as an expression in powers of ψ and ψ^*. The lowest order term has the form

$$\mathbf{j}(\mathbf{r}) = \int \mathbf{K}(\mathbf{r},\mathbf{r}',\mathbf{r}'')\psi^*(\mathbf{r}')\psi(\mathbf{r}'')dr'dr''. \tag{5.1}$$

If ψ is assumed to have a linear variation near $\mathbf{r}$, this reduces to the Ginzburg–Landau equations. A more appropriate approximation for barriers is to assume that ψ is slowly varying on each side of the barrier but changes suddenly as one crosses the barrier; this assumption leads to Eq. (1.3).

An important feature of (5.1) is that the relation between $\mathbf{j}$ and ψ is nonlocal. This leads to qualitatively different results from those of Bardeen,[28] who assumed a local relation. Detailed consideration of (5.1) shows that coupling is not dependent on an interaction in the barrier, but on the ability of electrons to propagate elastically through the barrier (as described by the single-electron Green's function). It is further necessary that time-reversal symmetry should apply to the propagation process. Finally, from (5.1) one can deduce that repulsive interactions in the barrier are, as one might expect, detrimental to coupling.

VI. CONCLUSION

With the exception of the last section, my aim in this paper has been to show that just as in the days before the BCS theory one knew a great deal about the behavior of superconductors purely on the basis of phenomenological theories, one can similarly predict many varied types of behavior of barriers between superconductors on the basis of a few simple equations based on straightforward assumptions. These equations can be justified by the microscopic theory, and, as far as present experimental evidence goes, seem to give a substantially correct description of events.

[27] J. Bardeen (private communication).

[28] J. Bardeen, Phys. Rev. Letters **6**, 57 (1961) and **9**, 147 (1962).