Flux flow resistivity in type II superconductors: II. Theoretical discussion

PROC. PHYS. SOC., 1967, VOL. 91. PRINTED IN GREAT BRITAIN

Flux flow resistivity in type II superconductors
II. Theoretical discussion

W. F. VINEN and A. C. WARREN
Department of Physics, University of Birmingham
MS. received 5*th January* 1967, *in revised form* 13*th March* 1967

Abstract. Existing theories of flux flow resistivity in type II superconductors are reviewed, and an extension of the work of Nozières and Vinen is introduced, in which the motion of a flux line is related to certain relaxation times governing the rate of change of the superconducting order parameter. The various experimental results that we now have on flux line motion are compared with this extended theory.

1. Introduction

The theory of the motion of flux lines in type II superconductors has been discussed by Tinkham (1964), by van Vijfeijken and Niessen (1965 a, b), by Bardeen and Stephen (1965, to be referred to as BS) and by Nozières and Vinen (1966, to be referred to as NV). In the present paper we discuss the validity of these theories and introduce an extension of the work of NV, which we try to apply to both alloys and pure metals. We then use this extension as the basis for a discussion of the significance of the various experimental results that we now have on flux flow resistivity, including those presented in the preceding paper (Vinen and Warren 1967, to be referred to as I), and of the extent to which we understand them. It may be stated at this point that, although they are in a sense quite successful, the theories discussed in the present paper are very crude and therefore have severe limitations. As we shall see, we believe that they are useful, not so much perhaps in accounting directly for the experimental results, but rather in drawing attention to some of the basic physical problems involved in the construction of a proper theory.

Let us suppose that a single rectilinear flux line, carrying one quantum of flux, $\phi = hc/2e$, lies along the z axis, and that there is an applied flow of superfluid past the core of the vortex with a velocity, measured on the z axis in the absence of the vortex, equal to $\mathbf{v}_{s1}$ and directed along the x axis. This applied flow might be due, for example, to an arrangement of other vortices. Throughout the present paper we shall assume that the radius of the vortex core is small compared with the penetration depth ($\kappa \gg 1$), so that the applied flow may be regarded as uniform over the core region. We shall also assume that the situation is two-dimensional, so that $\mathbf{v}_{s1}$ does not vary with z. We shall measure all velocities in the lattice frame of reference. Under the influence of the velocity $\mathbf{v}_{s1}$, the flux line will move with velocity $\mathbf{v}_L$ in the xy plane, and we conveniently write the relationship between these two velocities in the general form (assumed to be linear)

$$\mathbf{v}_{s1} \times \boldsymbol{\phi} - \beta \mathbf{v}_L \times \boldsymbol{\phi} - \gamma \phi \mathbf{v}_L = 0. \tag{1.1}$$

The theory of flux line motion is required to predict the values of the constants β and γ.

Provided that shearing effects can be ignored (see I), the flux flow resistivity and Hall angle in the mixed state are related to β and γ by the relations

$$\rho = \frac{\gamma}{\beta^2+\gamma^2}\frac{B}{N_s e} \tag{1.2}$$

$$\tan\theta_H = \frac{\beta}{\gamma} \tag{1.3}$$

where $\mathbf{B}$ is the mean magnetic induction in the sample and N_s is the number of superconducting electrons per unit volume well away from the core of the vortex. These relations depend on the fact that the *measured* electric field in a type II superconductor is equal to $(-1/c)\mathbf{v}_L \times \mathbf{B}$ (see e.g. NV, appendix A), and they also depend on the assumption that the cores of the flux lines are not too close together (H not too close to H_{c2}). It should be emphasized that we are considering in the present paper situations in which there is no pinning of the flux lines.

2. Review of earlier theories of flux line motion

A detailed rigorous theory of flux line motion clearly represents a very difficult problem, since the motion involves a situation where the order parameter is varying rapidly in space and time, where there is current flow through this region of varying order parameter, and where relationships between various quantities involved (currents, fields, order parameter) must be non-local. It is therefore of interest to ask first whether there exist any general results that must apply to flux line motion, independently of the detailed structure of the core.

One such general result has been proposed recently by NV. This is that at least in a pure superconductor at $T = 0$ flux lines are subject to the classical Magnus effect, in the sense that their motion is governed by the formula

$$\mathbf{f}+\frac{N_s e}{c}(\mathbf{v}_{s1}-\mathbf{v}_L)\times\boldsymbol{\phi} = 0 \tag{2.1}$$

where $\mathbf{f}$ is the total force on the electrons in the neighbourhood of the core due to interaction with the lattice. Equation (2.1) was derived by considering the balance of forces on a volume of the electron fluid within which the flux line core is wholly contained and by assuming that the only forces acting are the force $\mathbf{f}$, electromagnetic forces, and forces due to fluid pressure. Evaluation of the force $\mathbf{f}$ requires, of course, a detailed theory.

Such detailed theories have so far been based only on *models* of the core of the flux line. The models are based on the assumption that the superfluid density at any point in the vortex depends only on the local value of the superfluid velocity. For the isolated vortex, without imposed currents, we then expect that, owing to depairing of the electrons, the superfluid density will vanish at a certain radius from the centre of the flux line core, and we denote this radius by ξ. Inside this radius the material is entirely normal. Outside this 'normal core' the superfluid density may be taken either to be constant (with a discontinuity at the core boundary) or, more realistically, to rise rapidly and gradually to its equilibrium value (the 'transition region'). These models are clearly unrealistic, but they may nevertheless contain the essential physics of the problem.

The most ambitious detailed theory to be proposed so far is that of BS, which takes into account a transition region of finite extent; it applies only to clean superconductors ($l \gg \xi$) at $T = 0$.† (l is the electron mean free path in the normal state.) It is found that, essentially owing to Bernoulli effects, the motion of a flux line in the presence of the velocity $\mathbf{v}_{s1}$ is accompanied by an electric field which is large in the region of the normal core; this field gives rise to a normal current through the core and the resulting forces and dissipation determine the precise motion of the line for the given $\mathbf{v}_{s1}$. The theory predicts that

$$\rho = \frac{\rho_n B}{H_{c2}}, \qquad \tan\theta_H = \omega_c(H_0)\tau.$$

ρ_n is the normal-state resistivity, $\omega_c(H_0)$ is the cyclotron frequency in the field in the core of the flux line, and τ is the normal-state electron relaxation time.

This theory has been criticized by NV, mainly because of a questionable assumption that within the region of the core there is local equilibrium of the electrons with the lattice, even though the electron mean free path is large compared with the scale of this region. To this point of difficulty, we should like to add two others.

(i) The BS theory does not satisfy equation (2.1) but instead the equation

$$\mathbf{f}-\frac{N_s e}{c}\mathbf{v}_L\times\boldsymbol{\phi} = 0. \tag{2.2}$$

† A similar, but less complete, theory has been proposed independently by Van Vijfeijken and Niessen (1965 a, b).

The origin of this discrepancy is as follows. When a flux line moves, there is a flow of electrons through the surface of the normal core. Since the electrons in the superconducting state must approach the core and leave it with angular momentum appropriate to the vortex motion, this flow must involve the generation in the neighbourhood of the core surface of a net force, which is easily seen to be given by

$$\mathbf{f}_{\text{int}} = \frac{N_s e}{2c}(\mathbf{v}_{sc} - \mathbf{v}_L) \times \boldsymbol{\phi} \tag{2.3}$$

where $-N_s e \mathbf{v}_{sc}$ represents the supercurrent (assumed uniform) flowing into the normal core (see NV, equation (13)). It is reasonable to assume that this force acts eventually on the lattice (either directly or through the normal electrons) and hence that it contributes to $\mathbf{f}$. According to BS, however, part of the force (2.3), viz. $(N_s e/2c)\mathbf{v}_{sc} \times \boldsymbol{\phi}$, does not in fact act in this way; we believe that to this extent their analysis must probably be wrong, although it is possible that the error leaves the predictions of the model unaltered. It should be noted that the extra force does indeed make up the difference between (2.2) and (2.1), since, in the BS model, the supercurrent entering the core is equal to twice the transport current $N_s e \mathbf{v}_{s1}$ (see BS, equation (3.18)); we remember that there is also a force on the lattice equal to $(-N_s e/2c)\mathbf{v}_L \times \boldsymbol{\phi}$ due to an electric field acting on the normal core electrons (see appendix 1, equation (A2)).

(ii) The flow pattern in the transition region of the BS model is such that the supercurrent and normal current are *separately* divergence-free, except on the surface of the normal core, and, as a result, the whole of the momentum loss (2.3) occurs on this surface. We feel that this situation is physically unreasonable. It is not clear whether this difficulty arises from an inherent deficiency in the model, or whether some of the equations used by BS are unrealistic. We certainly believe that equation (3.12) of BS is questionable, because, as we show in appendix 2, it appears to contain the *assumption* that the two currents are separately divergence-free. We note also that in the BS model the net current flowing at any point is simply the sum of the current due to the isolated vortex and the uniform transport current. Since, as we have seen, the supercurrent flowing *into the core* equals twice the transport current, this condition on the *net* current can be satisfied only by having a *counterflow* of the superfluid and the normal fluid in the parts of the transition region where the core boundary is largely normal to the transport current. We feel that this counterflow is also physically unreasonable. We believe indeed that a realistic treatment of the transition region is much more difficult than that given by BS.

In view of difficulties with the BS model, NV proposed a different model, again applicable only to a pure metal at $T = 0$. No attempt was made to treat a transition region, and the superfluid density was taken as changing discontinuously at the boundary of the normal core. If this model is treated with the BS assumption of local equilibrium with the lattice, and with the requirement that the normal core current equals the transport current, then it is necessary to assume, as was shown by NV, that during motion of the line there is a non-zero contact potential at the core boundary (see appendix 1). This contact potential gives rise to a force on the electrons, which must be balanced by a force from the lattice; a corresponding force exists in the BS model (when modified to be consistent with equation (2.3)), but it appears in this model as the contribution to (2.3) that arises from that part of the supercurrent entering the normal core which is in excess of the transport current.

NV suggested that it might be more realistic to assume not that there is local equilibrium with the lattice, but rather that there is no contact potential (it may be seen from the comments that we have just made that, in a more realistic model with a transition region of finite extent, this assumption of zero contact potential is probably equivalent to an assumption of no counterflow in the transition region). Unfortunately, it turns out that the problem is not then completely determined; it is still necessary to obtain a relationship between $\mathbf{v}_{s1}$ and $\mathbf{v}_{nc}$, where $\mathbf{v}_{nc}$ is the drift velocity of the electrons in the normal core. It is difficult to determine this relationship, and NV simply assumed, on the basis of a vague energy argument, that $\mathbf{v}_{s1} = \mathbf{v}_{nc}$. This is clearly unsatisfactory.

In the next section we shall present an extension of the work of NV, in which this unsatisfactory feature is circumvented. The relationship between $\mathbf{v}_{s1}$ and $\mathbf{v}_{nc}$ is still not determined *a priori*, but it is expressed in terms of certain relaxation times that can be associated with the motion of the flux line. We determine the values of these times by comparison with the experimental results, and then try to understand the significance of these values. We also try to extend the treatment to dirty systems.

Before the detailed BS theory was published, Tinkham (1964) proposed that the motion of a flux line might be determined in part by dissipation due to a relaxation process associated directly with the fact that the order parameter near the core of the flux lines is forced to change with time, and he was led to suggest that for a dirty superconductor a relaxation time, equal to ξ_0/v_F, was associated with this process (ξ_0 is the Pippard coherence length in the pure superconductor and v_F is the Fermi velocity). We shall see that the present analysis involves a relaxation process of the same type, and that it leads to the same relaxation time.

3. Extension of the analysis of NV

We use the model described in § 6 of NV, viz. one in which the flux line has a normal core of radius ξ, with superfluid of uniform density outside the core. We consider only the case $T = 0$. However, we do not assume that $\mathbf{v}_{nc} = \mathbf{v}_{s1}$; instead, as we have already explained, we shall derive a relationship between these quantities in terms of certain relaxation times, the values of which we shall determine by comparison with experiment. We continue to assume that there is no contact potential at the core boundary.† We carry out our analysis first for clean materials ($l \gg \xi_0$) and then for dirty materials ($l < \xi_0$).

3.1. *Clean materials*

We first calculate the electric field acting on the normal core electrons, and hence obtain a relationship between the velocities $\mathbf{v}_L$, $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$. To do this we make use of a Bernoulli equation applicable in the superfluid outside the core of the flux line; this equation takes the form

$$\mu - eV + \tfrac{1}{2}m(v_s - v_L)^2 = \text{constant} \tag{3.1}$$

as was shown in NV (equation (9)). e is the magnitude of the electronic charge, μ is the chemical potential per electron in the superfluid (excluding contributions from the electrostatic potential and from the kinetic energy of flow), $\mathbf{v}_s$ is the total superfluid velocity at any point, and V is an electrostatic potential, defined so that the total electric field at any point is given by

$$\mathbf{E} = -\frac{1}{c}\mathbf{v}_L \times \mathbf{H} - \operatorname{grad} V. \tag{3.2}$$

In the neighbourhood of a flux line the total superfluid velocity is given by (NV, § 2)

$$\mathbf{v}_s(\mathbf{r}) = \mathbf{v}_{s0}(\mathbf{r}) + \mathbf{v}_{s1} + \mathbf{v}_b(\mathbf{r}). \tag{3.3}$$

$\mathbf{v}_{s0}(\mathbf{r})$ is the circular velocity due to the flux line itself, and $\mathbf{v}_b(\mathbf{r})$ is a dipolar backflow, needed for charge conservation, given by

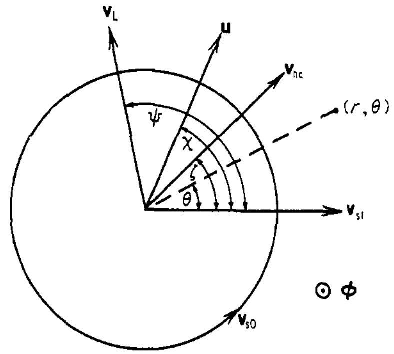
$$\mathbf{v}_b(\mathbf{r}) = \operatorname{grad}\left\{(\mathbf{v}_{s1} - \mathbf{v}_{ns}) \cdot \mathbf{r}\frac{\xi^2}{r^2}\right\}. \tag{3.4}$$

Using (3.3) and (3.4), we may express $\mathbf{v}_s(\mathbf{r})$ in terms of its components in cylindrical polar coordinates:

$$\left.\begin{aligned} v_{s\theta} &= \frac{\hbar}{2mr} - v_{s1}\sin\theta - \frac{\xi^2}{r^2}\{v_{s1}\sin\theta + v_{nc}\sin(\zeta - \theta)\} \\ v_{sr} &= v_{s1}\cos\theta - \frac{\xi^2}{r^2}\{v_{s1}\cos\theta - v_{nc}\cos(\zeta - \theta)\} \end{aligned}\right\}. \tag{3.5}$$

The notation is explained in the figure. From (3.5) we can calculate for any point on the

† More precisely we assume that the transport current and the consequent motion of the flux line lead to no *change* in any contact potential at the core boundary.



Sketch defining the angles θ, ξ, χ, ψ.

core boundary ($r = \xi$) the value of $(\mathbf{v}_s - \mathbf{v}_L)^2$, which we need to obtain the electric field in the core, and the value of $\mathbf{v}_s^2$, which we shall need later. To first order in the velocities $\mathbf{v}_{s1}$, $\mathbf{v}_{nc}$, $\mathbf{v}_L$ (v_{s1}, v_{nc}, $v_L \ll \hbar/2m\xi$), we find that

$$(\mathbf{v}_s - \mathbf{v}_L)^2 = v_{s0}^2(\xi) - \frac{\hbar}{m\xi}\{2v_{s1}\sin\theta + v_{nc}\sin(\zeta-\theta) + v_L\sin(\psi-\theta)\} \qquad (r = \xi) \tag{3.6a}$$

and

$$v_s^2 = v_{s0}^2(\xi) - \frac{\hbar}{m\xi}\{2v_{s1}\sin\theta + v_{nc}\sin(\zeta-\theta)\} \qquad (r = \xi). \tag{3.6b}$$

The potential V at any point just outside the core of the flux line is now given by equations (3.1) and (3.6*a*). Let us assume for the moment that μ does not depend on position. We also assume, as we have explained, that there is no contact potential at the core boundary. We see then that the potential V implies the existence inside the normal core of a contribution to the electric field which is uniform and which may be conveniently written

$$\mathbf{E}_c = \frac{1}{2\pi c\xi^2}(\mathbf{v}_L - 2\mathbf{v}_{s1} + \mathbf{v}_{nc}) \times \boldsymbol{\phi}. \tag{3.7}$$

The normal core electrons experience three driving forces: that due to the field $\mathbf{E}_c$, that due to the contribution $(-1/c)\mathbf{v}_L \times \mathbf{H}$ to the electric field (equation (3.2)), and that due to the interaction of the electric current in the core and the magnetic field in the core. The second and third of these forces may be shown to be small to order ξ^2/λ^2 (see NV, § 2), and we shall neglect them. We find then that the velocity $\mathbf{v}_{nc}$ is given by

$$\mathbf{v}_{nc} = -\frac{e\tau}{m}\mathbf{E}_c = -\frac{x}{\phi}(\mathbf{v}_L - 2\mathbf{v}_{s1} + \mathbf{v}_{nc}) \times \boldsymbol{\phi} \tag{3.8}$$

where

$$x = \frac{e\tau\phi}{2\pi mc\xi^2}.$$

This is the required relation between $\mathbf{v}_{nc}$, $\mathbf{v}_{s1}$ and $\mathbf{v}_L$. If we follow BS and assume that

$$\xi^2 = \frac{\hbar c}{2eH_{c2}} \tag{3.9}$$

then we find that

$$x = \frac{e\tau}{mc}H_{c2} = \omega_c(H_{c2})\tau. \tag{3.10}$$

BS give good reasons for believing the relation (3.9), but we cannot be sure that it is correct. However, provided it is incorrect by less than a factor of 2, our conclusions are not substantially altered.

In deriving equation (3 8) we assumed that the chemical potential μ does not depend on position. We recall that at $T = 0$ this assumption implies that the pressure is independent of position. It seems likely that this assumption is in fact justified. For let us suppose that we have a free electron gas, and that we make a small local change in its density (the total electron density, and the length scale involved in the density change, being comparable with those in the present problem); then, as is easily verified, the forces due to the potential set up by the resulting space charge greatly exceed those due to the resulting pressure gradient, and we believe that the same result holds for a superconducting electron gas. But the assumption may in fact be unnecessary. For we may say that the force acting on an electron is in general equal to $-(\nabla\mu - e\nabla V)$ instead of merely $-e\nabla V$, and a plausible generalization of the condition of zero contact potential is that $\mu - eV$ be continuous across the core boundary The result (3.8) then remains true. But this generalization does involve the difficulty that μ may not be locally defined in the core region, so we prefer on the whole to keep to the assumption that the force on the electrons due to the electric field is the only force of its type that acts.

We again remember that, as electrons move into and out of the core (depairing and pairing), momentum (associated with the circulating current in the vortex) must be lost and gained; in the present case there will be a corresponding force on the lattice that is given by (cf. equation (2.3))

$$\mathbf{f}_{\mathrm{int}} = +\frac{Ne}{2c}(\mathbf{v}_{\mathrm{nc}} - \mathbf{v}_{\mathrm{L}}) \times \boldsymbol{\phi}. \tag{3.11}$$

We have now put $N_s = N$, the total number of conduction electrons per unit volume, as is appropriate for a pure superconductor at $T = 0$. As in NV we assume that the force (3.11) is transferred ultimately to the lattice, and that this transfer is effected within a distance from the core boundary that is small compared with ξ. By adding together the force (3.11) and the electrostatic force $-\pi Ne\xi^2\mathbf{E}_c$, acting on the core electrons, we find that the total 'frictional' force acting between the lattice and the electrons in the region of the core is equal to the Magnus force $(Ne/c)(\mathbf{v}_L - \mathbf{v}_{s1}) \times \boldsymbol{\phi}$, in agreement with the general argument of NV†. The validity of the assumption that the effects of force (3.11) are localized in the region of the core boundary is, of course, questionable, especially when $l \gg \xi$, as was emphasized in NV; we shall return to this point in § 5.

As yet the motion of the flux line is not completely determined. We still need the relation between $\mathbf{v}_{\mathrm{nc}}$ and $\mathbf{v}_{s1}$, which we now derive.

Let us suppose first that we have an isolated vortex, with no superimposed currents. The core boundary will be at the points where the magnitude of the superfluid velocity is $\hbar/2m\xi$. The superfluid velocity at the core boundary is always tangential to this boundary, so that, as we expect, no normal current through the core is required for charge conservation. Now let us suppose that we artificially displace the core boundary, relative to the superfluid flow pattern, so that each point on the boundary moves a small constant distance $\mathbf{D}$. If we view the superfluid flow pattern from the centre 0 of the displaced core, we now see a pattern formed by adding together a vortex centred on 0 and an extra velocity field given by

$$\mathbf{v}_2 = (\mathbf{D} \cdot \mathrm{grad})\mathbf{v}_{s0}. \tag{3.12}$$

If we remember that near the core of a flux line $\mathbf{v}_{s0}$ is irrotational, we can write

$$\mathbf{v}_2 = \mathrm{grad}(\mathbf{D} \cdot \mathbf{v}_{s0})$$

† It should be noted that in NV, § 6, this part of the argument was in effect inverted the force acting on the core electrons was deduced by subtracting the force (3.11) from the total Magnus force, the existence of the Magnus force having been previously deduced from a more general argument. Our present argument is perhaps therefore very slightly weaker than that in NV, but we have made the change in order to be able to extend the treatment to a dirty system, where, as we shall see, the idea of the Magnus force appears not to apply.

which in turn can be written

$$\mathbf{v}_2 = -\operatorname{grad}\left(\mathbf{u}\,.\,\mathbf{r}\frac{\xi^2}{r^2}\right)$$

where the velocity $\mathbf{u}$ is given by

$$\mathbf{D} = \frac{2m\xi^2}{\hbar\phi}(\mathbf{u}\times\boldsymbol{\phi}). \tag{3.13}$$

We see that the extra velocity $\mathbf{v}_2$ takes the form of a dipolar backflow, and that its existence therefore implies a drift velocity through the normal core equal to $\mathbf{u}$ (compare with equation (3.4); we assume that the normal current is spatially uniform). Furthermore, the magnitude of the superfluid velocity at the core boundary is no longer equal to $\hbar/2m\xi$; as we see from a special case of (3.6*b*), $v_s{}^2$ is changed by an amount equal to

$$-\frac{\hbar}{m\xi}u\sin(\chi-\theta)$$

to first order in $\mathbf{u}$ (χ is defined in the figure).

We may generalize this picture by adding to the flow, both inside and outside the core, the uniform velocity $\mathbf{v}_{s1}$. The drift velocity in the normal core ($\mathbf{v}_{nc}$) is then equal to $\mathbf{u}+\mathbf{v}_{s1}$, while $v_s{}^2$ just outside the core differs from $\hbar^2/4m^2\xi^2$ by an amount equal to

$$-\frac{\hbar}{m\xi}\{u\sin(\chi-\theta)+v_{s1}\sin\theta\}.$$

We see therefore that the relationship between $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$ can be expressed both in terms of an effective displacement $\mathbf{D}$ of the core boundary and in terms of the magnitude of the superfluid velocity just outside the core. This remains true even when the core is continuously moving, provided we think about the instantaneous velocity fields.

Let us now apply these ideas to the determination of the relation between $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$ in the moving flux line. We shall make plausible assumptions about the position taken by the core boundary, and these will lead us to possible relationships between $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$.

We might first assume that the core boundary appears at points where the magnitude of the superfluid velocity *relative to the lattice* equals the critical value $\hbar/2m\xi$. We see that this leads immediately to the relation $\mathbf{u} = \mathbf{v}_{s1}$, i.e. $\mathbf{v}_{nc} = 2\mathbf{v}_{s1}$, and to the displacement $(2m\xi^2/\hbar\phi)(\mathbf{v}_{s1}\times\boldsymbol{\phi})$. We shall refer to this situation as one of 'equilibrium in the lattice frame'.

However, in the presence of the superimposed flow and when the flux line is moving, we cannot necessarily assume that this particular 'equilibrium' situation does in fact obtain. It seems likely that relaxation processes will lead to a further displacement of the core boundary: we shall make the reasonable assumption that this displacement contains terms that are proportional to the velocities $\mathbf{v}_L$ and $\mathbf{v}_{nc}$, and so can be written

$$\mathbf{D}_1 = -\tau_1{}'\mathbf{v}_L+\tau_1{}''\mathbf{v}_{nc} \tag{3.14}$$

where $\tau_1{}'$ and $\tau_1{}''$ are undetermined relaxation times. The time $\tau_1{}'$ is a relaxation time governing the rate at which the order parameter at any point returns to its equilibrium value, and it is therefore essentially the same as that introduced in connection with the same problem by Tinkham. The displacement (3.14) modifies the relation between $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$ which now becomes

$$\mathbf{v}_{nc}-2\mathbf{v}_{s1} = -\frac{\hbar}{2m\xi^2\phi}\mathbf{D}_1\times\boldsymbol{\phi} = -\frac{x}{\phi\tau}\mathbf{D}_1\times\boldsymbol{\phi}. \tag{3.15}$$

This is one form of the required relationship between $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$.

In the discussion that we have just presented, we assumed that the equilibrium position of the core boundary was determined by the condition that the magnitude of the superfluid velocity at the core boundary, *relative to the lattice*, be equal to the critical value $\hbar/2m\xi$. We must recognize that this assumption may not be justified, especially when $l \gg \xi_0$.

It might be the case that the critical superfluid velocity should instead be *relative to* $\mathbf{v}_{nc}$. In this case the equilibrium situation corresponds to $\mathbf{v}_{nc} = \mathbf{v}_{s1}$, and so to

$$\mathbf{D}_1 = \left(-\frac{\tau}{x\phi}\right)\mathbf{v}_{s1}\times\boldsymbol{\phi}.$$

We shall denote displacements from this position by $\mathbf{D}_n$, with corresponding relaxation times τ_n' and τ_n''. The correct view can be obtained only from a detailed microscopic analysis, which we do not attempt. In terms of $\mathbf{D}_n$, we can write

$$\mathbf{v}_{nc}-\mathbf{v}_{s1} = -\frac{x}{\phi\tau}\mathbf{D}_n\times\boldsymbol{\phi} \tag{3.16}$$

which is an alternative form of the relationship between $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$.

Combining equations (3.8), (3.14) and (3.15), we find that the equation of motion of the flux line has the form (1.1) with

$$\beta = \frac{x(x'+x'')}{2(1+xx'')}, \qquad \gamma = \frac{x+x'}{2(1+xx'')} \tag{3.17}$$

where $x' = (\tau_1'/\tau)x$ and $x'' = (\tau_1''/\tau)x$. In no sample so far studied (except those of Druyvesteyn *et al.* (1966)) has x exceeded a value equal to about 0·3, and we shall assume also that neither x' nor x'' exceeds this value. To a fairly good approximation, we may then neglect terms quadratic in x, x', x'' in comparison with unity. We find with this approximation that the flux flow resistivity ρ and the Hall angle θ_H are given by

$$\rho = \frac{2}{x+x'}\frac{B}{Ne} \tag{3.18}$$

$$\tan\theta_H = \frac{x(x'+x'')}{x+x'}. \tag{3.19}$$

3.2. *Dirty materials*

As mentioned in NV, the equation of motion for superfluid flow in dirty materials is not yet known with certainty. However, a form for this equation that is at least plausible has been suggested recently by Rickayzen (1966), and we shall use this form in the present discussion. We shall continue to assume that the superfluid density changes discontinuously at the boundary of the normal core and is constant outside the normal core; as pointed out by BS, this assumption is likely to be even less good for an alloy than for a pure metal.

For the case $T = 0$, to which we shall continue to confine our attention, the form suggested by Rickayzen is†

$$\frac{\partial\mathbf{v}_s}{\partial t} = -\nabla\left(\frac{\mu}{m}+\frac{1}{2}\frac{N_s}{N}v_s^2\right)-\frac{e}{m}\mathbf{E}. \tag{3.20}$$

The superfluid velocity $\mathbf{v}_s$ is defined in terms of the phase S of the order parameter by

$$\mathbf{v}_s = \frac{\hbar}{2m}\nabla S+\frac{e}{mc}\mathbf{A}; \tag{3.21}$$

μ is again the chemical potential per electron (excluding the electrostatic potential energy and the kinetic energy of flow); N_s/N is the effective fraction of superconducting electrons. For the very dirty case, $l \ll \xi$, at $T = 0$, this fraction is given approximately by

$$\frac{N_s}{N} = \frac{l}{\xi_0} \tag{3.22}$$

† It should be noted that equation (3.20) is not of the form that one expects for a single fluid in classical hydrodynamics, and we cannot therefore any longer expect to be able to discuss vortex motion from the point of view of a Magnus effect.

where l is the electron mean free path in the normal state, and ξ_0 is the Pippard coherence length in the pure metal. London's equation remains of the form

$$\operatorname{curl} \mathbf{v}_s - \frac{e}{mc}\mathbf{H} = 0 \tag{3.23}$$

the supercurrent being given by $J_s = -N_s e \mathbf{v}_s$.

We can now repeat the analysis required to obtain the equation of motion of a flux line, but with the modified equation of superfluid motion. We first derive the modified Bernoulli equation, and then repeat the type of analysis given in § 3.1 above. As in NV, we work throughout in the lattice frame of reference.

If variations in magnetic field are due entirely to flux line motion, one of Maxwell's equations takes the form

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{c}(\mathbf{v}_L \cdot \nabla)\mathbf{H}. \tag{3.24}$$

It follows (as in § 3.1 and in NV) that the electric field may be written in the form (3.2). We now add to each side of (3.20) the term $(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s$, and so obtain

$$\begin{aligned}\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s &= \tfrac{1}{2}\nabla(\mathbf{v}_s - \mathbf{v}_L)^2 - (\mathbf{v}_s - \mathbf{v}_L)\times \operatorname{curl}\mathbf{v}_s \\ &= \nabla\left\{\tfrac{1}{2}\left(1-\frac{N_s}{N}\right)v_s^2 - \frac{\mu}{m}\right\} - \mathbf{v}_s \times \operatorname{curl}\mathbf{v}_s - \frac{e}{m}\mathbf{E}. \end{aligned}\tag{3.25}$$

We have used the fact that changes of $\mathbf{v}_s$ in time are due to flux line motion, so that $\partial \mathbf{v}_s/\partial t = -(\mathbf{v}_L \cdot \nabla)\mathbf{v}_s$. Hence, using equations (3.23) and (3.25), we obtain the Bernoulli equation

$$\mu - eV + \tfrac{1}{2}m(\mathbf{v}_s - \mathbf{v}_L)^2 - \tfrac{1}{2}m\left(1-\frac{N_s}{N}\right)v_s^2 = \text{constant}. \tag{3.26}$$

The analysis of the flux line motion now follows exactly the same pattern as that in § 3.1, and we shall simply quote the results. The only point to which we need draw attention is that, although the supercurrent is carried effectively by only a fraction N_s/N of the electrons, the normal current in the core is carried by the whole fraction (there is therefore a discontinuity in the normal component of the velocity at the core boundary).

The electric field acting on the normal core electrons, analogous to that in equation (3.7), is given by (again for $\mathbf{v}_L$, $\mathbf{v}_{s1}$, $\mathbf{v}_{nc} \ll \hbar/2m\xi$)

$$\mathbf{E}_c = \frac{1}{2\pi c\xi^2}\left(\mathbf{v}_L - \frac{2N_s}{N}\mathbf{v}_{s1} + \mathbf{v}_{nc}\right)\times \boldsymbol{\phi} \tag{3.27}$$

(we again assume for simplicity that μ does not depend on position). Equation (3.8) is therefore replaced by

$$\mathbf{v}_{nc} = -\frac{x}{\phi}\left(\mathbf{v}_L - \frac{2N_s}{N}\mathbf{v}_{s1} + \mathbf{v}_{nc}\right)\times \boldsymbol{\phi}. \tag{3.28}$$

Again we express the relationship between $\mathbf{v}_{nc}$ and $\mathbf{v}_{s1}$ in terms of relaxation times τ_1' and τ_1'': equation (3.15) becomes

$$\frac{N}{N_s}\mathbf{v}_{nc} - 2\mathbf{v}_{s1} = -\frac{x}{\phi\tau}\mathbf{D}_1 \times \boldsymbol{\phi} \tag{3.29}$$

while we take the form (3.14) to be unchanged, although the actual values of τ_1' and τ_1'' may be different.

Combining equations (3.28), (3.29) and (3.14), we find that the equation of motion of the flux line is still of the form (1.1), but with

$$\beta = \frac{x(x'+x'')}{2\{1+(N_s/N)xx''\}}, \qquad \gamma = \frac{N}{N_s}\frac{\{x+(N_s/N)x'\}}{2\{1+(N_s/N)xx''\}}. \tag{3.30}$$

In dirty materials, $x \ll 1$, and we assume that the same is true for x' and x''. We may therefore neglect terms quadratic in x, x', x'' in comparison with unity. We then find the following expressions for the flux flow resistivity and Hall angle:

$$\rho = \frac{2}{x+(N_s/N)x'}\frac{B}{Ne} \tag{3.31}$$

$$\tan\theta_H = \frac{N_s}{N}\frac{x(x'+x'')}{x+(N_s/N)x'}. \tag{3.32}$$

4. Comparison with experiment

As we have seen, we find from experiment, at $T = 0$, for both pure materials and dirty materials that (in the notation of I) $\rho^* = B^*$, i.e.

$$\rho = \frac{1}{x}\frac{B}{Ne}. \tag{4.1}$$

Let us assume first that the position of the core boundary is determined on the basis of 'equilibrium in the lattice frame'. It follows from equation (3.18) and (3.31) that we must take

$$\tau_1' = \tau \qquad \text{in clean materials} \tag{4.2}$$

$$\tau_1' = \frac{N}{N_s}\tau \qquad \text{in dirty materials.} \tag{4.3}$$

The form (4.3) may be rewritten

$$\tau_1' = \frac{\xi_0\tau}{l} = \frac{\xi_0}{v_F} = \frac{\hbar}{\pi\Delta} \tag{4.4}$$

where v_F is the Fermi velocity, Δ is the superconducting energy gap, and where we have made use of equation (3.22). Thus it appears that, if our analysis is correct, the relaxation time τ_1' has to be taken in general to be either the electron–lattice relaxation time in the normal state, or the reciprocal of the gap frequency, whichever is the larger. This statement seems physically reasonable. If it is the superfluid velocity *relative to the lattice* that is involved in depairing, then it is likely that impurity scattering, and hence τ, will also be involved; it also seems reasonable that the order parameter cannot change at a rate greater than the gap frequency. But a proper microscopic analysis would be required to check these ideas. We note that the relaxation time (4.4) is essentially the same as that suggested by Tinkham.

Alternatively, we may assume that the position of the core boundary is determined on the basis of equilibrium in the normal electron frame. We then find that we must take τ_n' to be small compared with τ in all circumstances. In the case of a clean material this statement appears to be as physically reasonable as (4 2), for, if the lattice is not involved in depairing, τ_n' might well be determined by the time taken for a normal electron to cross the core which is the same as the reciprocal of the gap frequency in a clean material and which is much smaller than τ; but the statement appears to be less reasonable for a dirty material since τ_n' would then have to be less than the reciprocal of the gap frequency. It is interesting to note that, if the position of the core boundary is determined on the basis of equilibrium in the lattice frame, then relaxation processes appear to displace it to the position corresponding to equilibrium in the normal electron frame, while, if the position of the core boundary is determined on the basis of equilibrium in the normal electron frame, then relaxation processes do not displace it appreciably Thus the observed resistivity may actually be *independent* of the particular frame in which equilibrium exists, and this observation may turn out to be important in accounting for the fact that Kim's empirical formula is so widely applicable.

The time τ_1'' must be determined from the observed Hall angle θ_H. As yet, experimental measurement of θ_H in a clean material (niobium) has proved difficult. In the conventional d.c. measurements of Reed *et al.* (1965) the observed θ_H was strongly current-dependent, owing presumably to defects in the sample, and the same appears to be the case in the helicon measurements of Maxfield and Johnson (1966). However, in the very recent helicon measurements of Druyvesteyn *et al.* (1966) the current dependence was not found, although it was still present in d.c. measurements on the same samples. If we accept these most recent helicon measurements as giving the true Hall angle, we find that for pure niobium (with $\omega_c(H_{c2})\tau$ ranging from about 0·5 to about 1) $\tan\theta_H \simeq x$, and hence that either

$$\tau_l'' \simeq \tau \tag{4.5}$$

or

$$\tau_n'' \simeq 0 \tag{4.6}$$

where we have used the accurate versions of (3.18) and (3.19). Like those for τ_l' and τ_n', these results seem physically reasonable. There is, however, some uncertainty about this Hall angle, since the value of the resistivity derived from the helicon measurements appears not to agree with (4.1). We must also remember that in niobium the value of κ is small, so that, contrary to our assumptions, the core magnetic field is likely to contribute appreciably to the Hall angle.

Satisfactory Hall angle measurements do exist for one particular dirty superconductor, namely the alloy NbTa (Staas *et al.* 1965), for which $l/\xi \sim 0{\cdot}79$. For fields that are well below H_{c2} (to which the present theory is confined), these measurements indicate a value of $\tan\theta_H$ equal to about $2{\cdot}4x$. This can be accounted for by taking a suitable value of τ_l'', but it is not clear what meaning can be attached to this value.

5. Discussion

We must emphasize that the discussion in the present paper is based on a very crude model, which can be analysed with ease only at $T = 0$ and for $\kappa \gg 1$, and that it depends rather critically both on the approximate validity of equation (3.9) and on the absence of any contact potential at the core boundary. There is also the difficulty, already mentioned, connected with the transfer of the force (3.11) to the lattice. We have assumed that this transfer takes place in the immediate neighbourhood of the core boundary. This is a reasonable assumption in the case of the dirty system, where the mean free path of the normal core electrons is much less than the core diameter, but it is less reasonable in the clean case. Indeed, in the clean case it might well be that we ought to imagine that the transfer of the force to the lattice is uniformly distributed over the core region, and that the force therefore acts as an extra driving force on the core electrons. We can easily work out the consequences of this latter assumption: the analysis is a straightforward modification of that given in § 3.1. We find that to obtain agreement with experiment we have to take $\tau_n' = \tau_n'' = -\tau$. This is unreasonable. However, if we also modify the relation (3.9) by adding a factor of two on the right-hand side, we find that we can obtain agreement by taking $\tau_n' = \tau_n'' = 0$, as in § 4. Thus the difficulty over the force (3.11) can in a sense be avoided, although an ambiguity remains.

The crude nature of the model that we are using is bound to make one uneasy, as is the fact that we have fed into our calculations assumptions which, although plausible, have certainly not been properly justified, and the possibility cannot be ruled out that the success of the model is fortuitous. The sceptic may therefore object that the model is of little value. In one sense we are inclined to agree with this view, and to agree therefore that we may as yet have little real understanding of the motion of flux lines in superconductors. However, we also feel that the model may help to fix attention on the basic physical problems that lie behind a real understanding of this motion, and that it may therefore act as a guide in the construction of a proper theory†. But even this feeling may may prove too optimistic. For the proper theory may circumvent the kinds of problem

† After the present paper was written, the authors learned of two interesting attempts to construct a proper theory one by Schmid (1966), the other by Kulik (1966).

that arise in the model calculation. The empirical formula of Kim for the flux flow resistivity is after all such a simple result, and so widely applicable, that it may well be the consequence of some very general principle, which is quite independent of the detailed processes occurring in the neighbourhood of the core of the flux line. But we have been unable to find such a principle.

Acknowledgments

We are grateful to Professor J. Bardeen for many helpful discussions, and to the referee for encouraging us to present our work in greater detail than was originally intended.

Appendix 1

We give here a simple derivation of the BS equation of motion for a flux line, based on the type of approach used in the present paper. We assume that the superconductor is pure, and we continue to assume that the magnetic field in the core of the flux line can be neglected.

According to BS there is in effect a contact potential across the core boundary. This potential is a consequence of the assumption that there is local equilibrium of the electrons with the lattice, i.e. that the value of the total potential, $\mu - eV + \frac{1}{2}mv_s^2$, in the lattice frame just outside the core be equal to the value of $\mu - eV$ just inside the core. We continue to assume for simplicity that μ is everywhere the same. Then the value of the contact potential is seen to be

$$\Delta V = \frac{m}{2e} v_{sc}^2 \tag{A1}$$

where $\mathbf{v}_{sc}$ is the total superfluid velocity at the core boundary (assumed sharp). It follows that the electric field in the core is given by (v_L, v_{s1}, $v_{nc} \ll 1$)

$$\mathbf{E}_c = \frac{1}{2\pi c \xi^2} \mathbf{v}_L \times \boldsymbol{\phi} \tag{A2}$$

instead of by (3.7). Hence we find for the velocity $\mathbf{v}_{nc}$

$$\mathbf{v}_{nc} = -\frac{x}{\phi} \mathbf{v}_L \times \boldsymbol{\phi}. \tag{A3}$$

Like us, BS allow a displacement of the boundary of the moving core, owing to relaxation, but in effect they simply assume that (in our notation)

$$\mathbf{D}_1 = -\tau \mathbf{v}_L. \tag{A4}$$

Hence, using (3.15) and (A3) we find that

$$\mathbf{v}_{nc} = \mathbf{v}_{s1}. \tag{A5}$$

Therefore, from (A3) and (A4), we find that the values of β and γ in the equation of motion (1.1) are given by

$$\beta = 0, \qquad \gamma = x \tag{A6}$$

instead of by (3.17). This is the BS result, in the limit when the magnetic field in the core of the flux line can be ignored.

We note that the total force acting between the normal core electrons and the lattice is now equal to $(Ne/2c)\mathbf{v}_L \times \boldsymbol{\phi}$. In order that equation (2.1) can be satisfied, an additional force, equal to $(Ne/2c)(\mathbf{v}_L - 2\mathbf{v}_{s1}) \times \boldsymbol{\phi}$, must be assumed to act between the electrons and the lattice at the core boundary. Difficulties associated with this extra force have already been mentioned.

Appendix 2

In this appendix we attempt to derive equation (3.12) of BS. Essentially this equation has the form

$$\frac{\partial \mathbf{J}_s}{\partial t} + \frac{1}{\tau} \mathbf{J}_n = \mathbf{f} \tag{A7}$$

where $\mathbf{J}_s$ and $\mathbf{J}_n$ are the mass current densities associated with the superfluid and normal fluid, and τ is the relaxation time for electrons in the normal state; the force $\mathbf{f}$ is given by

$$\mathbf{f} = -\nabla\{F(\mathbf{v}_s)\} - Ne\mathbf{E} \tag{A8}$$

where $F(\mathbf{v}_s)$ is the free energy per unit volume associated with a superfluid velocity $\mathbf{v}_s$ (defined by our (3.21)), N is the number of electrons per unit volume, and $\mathbf{E}$ is the total electric field.

We start by assuming that the rate of change of the total mass current is given by

$$\frac{\partial J_i}{\partial t} = \frac{\partial J_{si}}{\partial t} + \frac{\partial J_{ni}}{\partial t} = -NeE_i - \frac{e}{m}(\mathbf{J}_n \times \mathbf{H})_i - \frac{e}{m}(\mathbf{J}_s \times \mathbf{H})_i - \frac{1}{\tau}J_{ni} - \frac{\partial}{\partial x_j}(v_{si}J_{sj} + v_{ni}J_{nj}) \tag{A9}$$

where the last term is the divergence of the total momentum flux density tensor, and $\mathbf{v}_n$ is the average velocity of the thermal excitations in the electron system. This equation seems reasonable except for the relaxation term $-J_{ni}/\tau$, where we have assumed, without justification, that a single relaxation time is involved, equal to that in the normal state. From (A9) we find that

$$\frac{\partial \mathbf{J}_s}{\partial t} + \frac{\partial \mathbf{J}_n}{\partial t} = -Ne\mathbf{E} - \frac{e}{m}(\mathbf{J}_s + \mathbf{J}_n) \times \mathbf{H} - \frac{1}{\tau}\mathbf{J}_n - \mathbf{v}_s \operatorname{div} \mathbf{J}_s - (\mathbf{J}_s \,.\, \nabla)\mathbf{v}_s - \mathbf{v}_n \operatorname{div} \mathbf{J}_n - (\mathbf{J}_n \,.\, \nabla)\mathbf{v}_n. \tag{A10}$$

In practice, the terms in $\partial \mathbf{J}_n/\partial t$, $\mathbf{J}_n \times \mathbf{H}$, $(\mathbf{J}_n . \nabla)\mathbf{v}_n$ are small, and we shall neglect them. Furthermore, the total current must be divergence-free, and hence

$$\frac{\partial \mathbf{J}_s}{\partial t} + \frac{1}{\tau}\mathbf{J}_n = -Ne\mathbf{E} - \frac{e}{m}\mathbf{J}_s \times \mathbf{H} - (\mathbf{v}_s - \mathbf{v}_n) \operatorname{div} \mathbf{J}_n - (\mathbf{J}_s \,.\, \nabla)\mathbf{v}_s. \tag{A11}$$

But we have from BS, equation (2.1),

$$\mathbf{J}_s = \frac{\partial F(\mathbf{v}_s)}{\partial \mathbf{v}_s} \tag{A12}$$

and hence

$$\begin{aligned} \nabla\{F(\mathbf{v}_s)\} &= (\mathbf{J}_s \,.\, \nabla)\mathbf{v}_s + \mathbf{J}_s \times \operatorname{curl} \mathbf{v}_s \\ &= (\mathbf{J}_s \,.\, \nabla)\mathbf{v}_s + \frac{e}{m}\mathbf{J}_s \times \mathbf{H}. \end{aligned} \tag{A13}$$

Therefore

$$\frac{\partial \mathbf{J}_s}{\partial t} + \frac{1}{\tau}\mathbf{J}_n = -Ne\mathbf{E} - \nabla\{F(\mathbf{v}_s)\} - (\mathbf{v}_s - \mathbf{v}_n) \operatorname{div} \mathbf{J}_s. \tag{A14}$$

This agrees with equation (A7) only if the supercurrent and normal current are separately divergence-free, and we believe therefore that (A7) can be valid only if this condition is satisfied (quite apart from the difficulty that the relaxation term is of uncertain validity).

References

Bardeen, J., and Stephen, M. J., 1965, *Phys. Rev.*, **140,** A1197–207.
Druyvesteyn, W. F., Van Gurp, G. J., and Greebe, C. A. A. J., 1966, *Phys. Letters*, **22,** 248–9.
Kulik, I. O., 1966, *Soviet Physics–JETP*, **23,** 1077–84.
Maxfield, B. W., and Johnson, E. F., 1966, *Phys. Rev. Letters*, **16,** 652–4.
Nozières, P., and Vinen, W. F., 1966, *Phil. Mag.*, **14,** 667–88.
Reed, W. A., Fawcett, E., and Kim, Y. B., 1965, *Phys. Rev. Letters*, **14,** 790–2.
Rickayzen, G., 1966, *Proc. Phys. Soc.*, **89,** 129–34.
Schmid, A., 1966, *Physics of Condensed Matter*, **5,** 302–17.
Staas, F. W., Niessen, A. K., and Druyvesteyn, W. F., 1965, *Phys. Letters*, **17,** 231–3.
Tinkham, M., 1964, *Phys. Rev. Letters*, **13,** 804–7.
Van Vijfeijken, A. G., and Niessen, A. K., 1965 a, *Phys. Letters*, **16,** 23–4.
—— 1965 b, *Philips Res. Rep.*, **20,** 505–27.
Vinen, W. F., and Warren, A. C , 1967, *Proc. Phys. Soc.*, **91,** 399–408.